K-theoretic Quasimap Wallcrossing

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$M =$ "space" (DM stacks, Artin stacks)

$[Y/\mathbb{C}^*]$

$K_0(M) = G(\mathrm{Coh}(M))$ ← proper pushforward

$K^0(M) = G(\mathrm{Vect}(M))$ ← pullback

$K_0(M)$ is $K^0(M)$

$K_0([Y/\mathbb{C}^*]) = K_0^{\mathbb{C}^*}(Y)$

$K^0([Y/\mathbb{C}^*]) = K^0_{\mathbb{C}^*}(Y)$

If M is proper, $p_*(F) = \chi(F)$ $\in K_0(pt) \cong \mathbb{Z}$ $p: M \to pt$.

$F \in K_0(M)$

K-theoretic GW-invariants, X = sm. proj orbifold

$\overline{M}_{g,n}(X,\beta)$ = moduli of stable map.

has a perfect obstruction theory.

$$\rightsquigarrow \mathcal{O}^{vir}_{\overline{M}_{g,n}(X,\beta)} \in K_0(\overline{M}_{g,n}(X,\beta))$$

$ev_i : \overline{M}_{g,n}(X,\beta) \longrightarrow \bar{I}X$ = rigidified inertia stack.

gerbe $\uparrow$

$\widetilde{M}_{g,n}(X,\beta) \longrightarrow IX$ = inertia stack.

$\alpha_1, \dots, \alpha_n \in K^0(\bar{I}X)$

$$\langle \alpha_1, \dots, \alpha_n \rangle^X_{g,n,k} = \chi\left(\mathcal{O}^{vir}_{\overline{M}_{g,n}(X,\beta)} \cdot \prod_{i=1}^{k} ev_i^*(\alpha_i)\right)$$

$$\in \mathbb{Z}$$

S_n - equivariant structure.

$$\overline{M}_{g,n}(X,\beta) \circlearrowleft S_n \qquad \mathcal{O}^{vir}_{\overline{M}} \text{ is equivariant}$$

$$\downarrow p$$

$$\overline{M}_{g,(n)}(X,\beta) = [\overline{M}_{g,n}(X,\beta)/S_n]$$

unordered markings.

$$p^*(\mathcal{O}^{vir}_{[\overline{M}/S_n]}) = \mathcal{O}^{vir}_{\overline{M}}$$

However $p_*(\mathcal{O}^{vir}_{\overline{M}}) \neq \mathcal{O}^{vir}_{[\overline{M}/S_n]}$

$\alpha \in K^0(\bar{I}X, \beta)$.

Then $\alpha^{\boxtimes n} \in K^0_{S_n}((\bar{I}X)^n)$

$\alpha = [E]$, $E^{\boxtimes n}$ S_n-equivariant.

$\alpha = [E_0] - [E_1]$. View $E_0 \oplus E_1$ as a super bundle,

Then $(E_0 \oplus E_1)^{\boxtimes n}$ S_n-equiv. twisted by sgn

$$\alpha^{\boxtimes n} := \left[\left((E_0 \oplus E_1)^{\boxtimes n}\right)^{even}\right] - \left[(\ldots)^{odd}\right]$$

$\overline{M}_{g,n}(X, \beta) \xrightarrow{ev} \prod_{i=1}^{n} \bar{I}X$ S_n-equivariant.

$$[\alpha, \dots, \alpha]_{g,n,\beta} = p_*\left(\mathcal{O}^{vir}_{\overline{M}} \cdot ev^*(\alpha^{\boxtimes n})\right) \qquad p: \overline{M} \to pt.$$

$$\in K_0^{S_n}(pt)$$

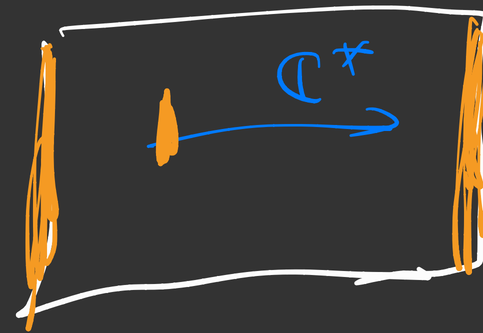
$$= R(S_n).$$

$$\langle \alpha, \dots, \alpha \rangle^{S_n} = S_n\text{-fixed part} \in K_0(pt) \cong \mathbb{Z}.$$

Take the state space $= K^0(\bar{I}X) \otimes \Lambda$

λ-ring

Master space technique



E vect bundle.

$$\lambda_t(E) = \sum t^k \overset{k}{\wedge} E, \qquad E \in K^0$$

K-theoretic Euler class

$$e^K(E) := \lambda_{-1}(E^\vee) = \sum (-1)^k \overset{k}{\wedge} E^\vee,$$

In particular, $E = L$. weight $\mathbb{C}^*$ weight. 1.

$$\lambda_{-1}^{\mathbb{C}^*}(L^\vee) = 1 - q^{-1}L^{-1}.$$

K-theoretic localization formula:

(Kiem – Savvas)

$\mathbb{C}^* \curvearrowright \mathcal{X}$, assume that $\boxed{\mathbb{C}^* \curvearrowright F_i = \text{fixed loci are trivial.}}$

achieved by $\mathbb{C}^* \xrightarrow{t \mapsto t^{u}} \mathbb{C}^*$

$\mathbb{C}^* \curvearrowright [\mathbb{A}^1/\mathbb{Z}_2]$ $\quad \mathbb{C}^* \curvearrowright B\mathbb{Z}_2$ nontrivial action,

$\boxed{t \cdot x = t^{1/2} x}$

$$\mathcal{O}^{vir}_{\mathcal{X}} = \sum_i \iota_* \frac{\mathcal{O}^{vir}_{F_i}}{\lambda_{-1}^{\mathbb{C}^*}\left(\left(N^{vir}_{F_i/\mathcal{X}}\right)^\vee\right)}$$

$$\in K_0^{\mathbb{C}^*}(\mathcal{X}) \otimes_{\mathbb{Q}[q,q^{-1}]} \mathbb{Q}(q), \qquad K(B\mathbb{C}^*) = \mathbb{Z}[q,q^{-1}]$$

Say $N^{vir}_{F_i/\mathcal{X}} = L$. ~~then~~

$$\frac{\mathcal{O}^{vir}_{F_i}}{1 - q^{-1}L^{-1}} \quad \text{v.s.} \quad \frac{[F]^{vir}}{z + c_1(L)}$$

K-~~theory~~ version $\qquad$ Chow version

Master space technique.

$$\mathbb{C}^* \circlearrowright \mathcal{X} \xrightarrow{p} \mathcal{Y}^{=pt.} \quad \mathbb{C}^*\text{-invariant.}$$

trivial $\mathbb{C}^*$-action.

$$p_*([\mathcal{O}_{\mathcal{X}}^{vir}]) \in K_0(\mathcal{Y}) \otimes \mathbb{Q}[q, q^{-1}]$$

So does the RHS.

$$\boxed{\alpha \in K^0_{\mathbb{C}^*}(\mathcal{X})}$$

Hence

$$\left(\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}\right)\left(\sum_i p_*\left(\frac{\mathcal{O}_{F_i}^{vir} \cdot \alpha}{\lambda_{-1}^{\mathbb{C}^*}\left(\left(N_{F_i/\mathcal{X}}^{vir}\right)^\vee\right)} \frac{dq}{q}\right)\right) = 0$$

If $N^{vir}_{F_i} = L$ with weight 1 action.

~~then~~

$$\left(\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}\right)\left(p_*\left(\frac{\mathcal{O}^{vir}_{F_i}\cdot\alpha}{1-q^{-1}L^{\vee}}\right)\frac{dq}{q}\right)$$

$$= -p_*\left(\alpha\cdot\mathcal{O}^{vir}_{F_i}\right)$$

ε-stable quasimaps:

$$X = [W^{ss}(\theta)/G] = \text{DM stack.}$$

W = affine schm. with mild sing.

char. of G.

reductive grp acting on W.

Assume. • $W^{ss}(\theta) = W^{s}(\theta)$.

• X is projective

Defn: A quasimap is

$$C \xrightarrow{u} [W/G] \supset X$$

twisted nodal curve (w/ marking).

rep'ble.

G-$P \to C$ + $P \times_G W \to C$, σ

$u^{-1}([W^{us}(\theta)/G])$ is discrete and disjoint with nodes and markings.

$\shortparallel$

base locus.

E.g. : $X = \mathbb{P}^N = \mathbb{C}^{N+1} /\!/ \mathbb{C}^*$

$\mathbb{P}^1 \dashrightarrow \mathbb{P}^N$

$\mathcal{L} = \mathcal{O}(d)$.

$\sigma_i = a_i y^d$

Then. $C \to [\mathbb{C}^{N+1}/\mathbb{C}^*]$

$$\Longleftrightarrow \left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ C \end{array} + \begin{array}{c} \mathcal{L}^{\oplus(N+1)} \\ \downarrow \uparrow \sigma \\ C \end{array} \right)$$

$(y=0)$ is a base point of length d.

$[a_0, \dots, a_d] \in \mathbb{P}^N$

base locus $= (\sigma_0 = \dots = \sigma_N = 0)$

is discrete + away from nodes + markings.

$$\boxed{\text{length}_x = \min\{\text{van. ord. } \sigma_i \text{ at } x\}}$$

$(F=0)$

$\cap\!\!|$

$\mathbb{C}^{N+1}$

$= W$

$$X = (\text{hypersurface } F=0) \subseteq \mathbb{P}^N$$

$$\| $$

$$W /\!/ \mathbb{C}^*$$

$$C \to [W/G] \supseteq X$$

$\iff$ $\begin{array}{c} \mathcal{L} \\ \downarrow \\ C \end{array}$ $+\ \sigma_0, \ldots, \sigma_N,$ s.t. $F(\sigma_0, \ldots, \sigma_N) = 0$

s.t. base locus condition . . .

$$I(Q,q) = 1 + (1-q^{-1}) \sum_{\beta>0} Q^{\beta} (\underline{\check{ev}_*})_* (\cdots)$$

$$\mu_\beta(q) = Q^{\beta}\text{-coeff in} \quad q^{1/r}$$

$$\left[(1-q) I(Q,q) - (1-q) \right]_+ .$$

moduli with trivialized gerbe marking

$$ev_* : QG_{0,1} \longrightarrow IX$$

not $\mathbb{C}^*$-invariant

$q^{1/r}$

$K^0(IX)$ is larger than $K^0(\bar{I}X)$

$$A_*(IX) \cong K^0(X)$$

$B\mathbb{Z}_2$. Wei Gu.

4 dim'l state space.

$$K^0(IB\mathbb{Z}_2) = K^0(B\mathbb{Z}_2) \oplus K^0(B\mathbb{Z}_2)$$